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点集合と超平面による切り口の次元について

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1. Introduction

Let $S \subset \mathbb{R}^3$ be the space described in K.Sitnikov[6] satisfying the relation $1 = \mu\dim S < \dim S = 2$, where $\mu\dim$ (resp. \dim) denotes the metric (resp. covering) dimension. As easily seen, the space S has a remarkable property that $\mu\dim(S \cap H) = \mu\dim S$ for every plane H in \mathbb{R}^3 . Motivated by this, we will be concerned with the problems whether there exists a point set X in Euclidean n -space \mathbb{R}^n satisfying (A) or both of the following two conditions:

- (A) $\mu\dim(X \cap H) = \mu\dim X$ for every hyperplane H in \mathbb{R}^n .
- (B) $\dim(X \cap H) = \dim X$ for every hyperplane H in \mathbb{R}^n .

Here by a *hyperplane* in \mathbb{R}^n , we mean an $(n-1)$ -dimensional affine subspace of \mathbb{R}^n .

The first result is the following which improves [2, Lemma 4]:

Theorem 1. *For arbitrary integers m and n with $0 \leq m \leq n-1 \geq 1$, there exists a point set X_m^n in \mathbb{R}^n such that*

- i) $\mu\dim X_m^n = m$ and $\dim X_m^n = \min\{2m, n-1\}$, and
- ii) $\mu\dim(X_m^n \cap H) = m$ for every hyperplane H in \mathbb{R}^n .

Let us note that if a non-empty space X in \mathbb{R}^n satisfies the condition (A), then necessarily $n \geq 2$ and $\dim X \leq n-1$. Moreover since $\dim X \leq 2\mu\dim X$ by a Katětov's inequality[4], the space X_m^n in Theorem 1 is one which admits the maximal difference between \dim and $\mu\dim$ among those spaces X in \mathbb{R}^n satisfying $\mu\dim X = m$ and the condition (A).

In contrast with Theorem 1, it will be shown that there exists Y_k^n in \mathbb{R}^n with $\mu\dim Y_k^n = \dim Y_k^n = k$ satisfying the condition (A) (and also (B)) for arbitrary integers n and k with $0 \leq k \leq n-1 \geq 1$ (Theorem 2).

Now suppose that a space X in \mathbb{R}^n satisfies both (A) and (B) with $\dim X = k$ and $\mu\dim X = m$. Then as above, it must be satisfied that $n \geq 2$ and $m \leq k \leq \min\{2m, n-1\}$, and also that either $k < n-1$ or $k = n-1 = m$; indeed, if $\dim X = n-1$, then $X \cap H$ must have non-empty interior in a hyperplane H by (B), which implies $\mu\dim X = n-1$.

The following is the main result which extends [3, Theorem]:

Main theorem. *Let n, m and k be arbitrary integers such that $0 \leq m \leq n - 1 \geq 1$ and $m \leq k \leq \min\{2m, n - 1\}$. Then there exists a point set $X_{m,k}^n$ in \mathbb{R}^n such that*

- i) $\mu\dim X_{m,k}^n = m$ and $\dim X_{m,k}^n = k$,
- ii) $\mu\dim(X_{m,k}^n \cap H) = m$ for every hyperplane H in \mathbb{R}^n , and
- iii) if either $k < n - 1$ or $k = n - 1 = m$, then $\dim(X_{m,k}^n \cap H) = k$ for every hyperplane H in \mathbb{R}^n .

2. Preliminaries

By \mathbb{I} we denote the closed interval $[-1, 1]$. Also, \mathbb{N} , \mathbb{Z} and \mathbb{Q} denote the sets of natural numbers, integers and rationals, respectively. Thus $\mathcal{F} = \{z + \mathbb{I}^n : z \in \mathbb{Z}^n\}$ is a collection of congruent n -cubes whose interiors cover \mathbb{R}^n . Similarly, $\mathcal{F}_i = \{(1/i)z + [0, 1/i]^n : z \in \mathbb{Z}^n\}$, $i \in \mathbb{N}$, is a cover of \mathbb{R}^n by n -cubes whose interiors are pairwise disjoint and sides are of length $1/i$. We set

$$F_i^{(j)} = \cup\{\tau^{(j)} : \tau \in \mathcal{F}_i\}, \quad 0 \leq j \leq n, \quad \text{where } \tau^{(j)} \text{ denotes the union of } j\text{-faces of } \tau.$$

Let $\alpha = \{a_i\}$ be a sequence of points in \mathbb{R}^n and m, n integers with $0 \leq m \leq n - 1$. Then we define

$$S_m^n(\alpha) = \mathbb{R}^n - \cup\{a_i + F_i^{(n-m-1)} : i \in \mathbb{N}\}.$$

Fact 1(cf. [2, Lemma 4].) $\mu\dim S_m^n(\alpha) = m$ for every sequence α of points in \mathbb{R}^n .

Indeed, for every i , $S_m^n(\alpha)$ admits a continuous map f onto the m -skeleton of the decomposition of \mathbb{R}^n by n -cubes which is dual to \mathcal{F}_i , satisfying $\|x - f(x)\| < \sqrt{n}/2i$ for every x . This implies $\mu\dim S_m^n(\alpha) \leq m$ (cf. [7, Corollary 2]), and the opposite inequality is obvious because $S_m^n(\alpha)$ contains a (rectilinear) m -simplex. The following is a special case of [8, Theorem 3].

Fact 2. *If a sequence $\alpha = \{a_i\}$ of points in \mathbb{R}^n satisfies the condition*

$$\dim((a_i + F_i^{(n-m-1)}) \cap (a_j + F_j^{(n-m-1)})) \leq k \quad \text{whenever } i \neq j,$$

then $\dim S_m^n(\alpha) \geq n - k - 2$.

For every finite set A of \mathbb{R}^n and every integer k with $0 \leq k \leq n - 1$, we set

$$A^{[k]} = \{[v_0, \dots, v_j] : v_0, \dots, v_j \in A, j \leq k\}$$

where $[v_0, \dots, v_j]$ denotes the plane(i.e., the affine subspace) determined by points v_0, \dots, v_j . Then we say that $p \in \mathbb{R}^n$ is in a *general position* (or *g.p.*, for short) *relative to* A , if $p \notin \cup A^{[n-1]}$.

We denote by $\pi_k : \mathbb{R}^n \rightarrow \mathbb{R}$, $1 \leq k \leq n$, the projection of \mathbb{R}^n into the k -th factor.

Fact 3.(cf.[2, Lemma 4]) *If $\alpha = \{a_i\}$ in \mathbb{R}^n satisfies the condition that*

$$(\pi_k(a_i) + (1/i)\mathbb{Z}) \cap (\pi_k(a_j) + (1/j)\mathbb{Z}) = \emptyset \quad \text{whenever } i \neq j,$$

for every $k = 1, \dots, n$, then $\dim S_m^n(\alpha) = \min\{2m, n - 1\}$.

This follows from Fact 2 and the fact that if α satisfies the condition in Fact 3, then

$$\dim((a_i + F_i^{(n-m-1)}) \cap (a_j + F_j^{(n-m-1)})) = \max\{n - 2m - 2, -1\}.$$

Sitnikov's space S cited above is of type $S_1^3(\alpha)$ with α satisfying the condition in Fact 3 for $(m, n) = (1, 3)$. Also the space X_m^n with $m > 0$, which will be given in the proof of Theorem 1, is of form $S_m^n(\alpha)$.

3. Point sets X_m^n and Y_k^n

For a sequence $\alpha = \{a_i\}$ of points in \mathbb{R}^n , we consider the condition:

(C_i) Every point $p \in (a_i + F_i^{(0)}) \cap \mathbb{I}^n$ is g.p.relative to $\cup\{(a_j + F_j^{(0)}) \cap \mathbb{I}^n : j < i\}$.

Lemma 1. *There exists a sequence $\alpha = \{a_i\}$ of points in \mathbb{Q}^n satisfying (C_i) for all $i \geq 2$.*

Let us note that (C_i) implies

$$(1) \quad (\pi_k(a_i) + (1/i)\mathbb{Z}) \cap (\pi_k(a_j) + (1/j)\mathbb{Z}) = \emptyset \quad \text{for every } j < i \text{ and } k = 1, \dots, n.$$

Since each $F_i^{(n-m-1)}$ can be expressed as the countable union of $(n - m - 1)$ -planes, there exist $(n - m - 1)$ -planes $B_{i,s}^{n-m-1}$ such that

$$(2) \quad a_i + F_i^{(n-m-1)} = \cup\{B_{i,s}^{n-m-1} : s \in \mathbb{N}\}, \quad i \in \mathbb{N}.$$

Lemma 2. *Suppose a sequence $\alpha = \{a_i\}$ of points in \mathbb{R}^n ($n \geq 2$) satisfies (C_i) for every $i \geq 2$. Then for every hyperplane H in \mathbb{R}^n with $H \cap \text{Int } \mathbb{I}^n \neq \emptyset$,*

$$\Lambda = \{i \in \mathbb{N} : B_{i,s}^{n-m-1} \cap \mathbb{I}^n \neq \emptyset, B_{i,s}^{n-m-1} \subset H \text{ for some } s \in \mathbb{N}\}$$

consists of at most n elements.

Lemma 3. *Let m and n be integers with $0 \leq m \leq n - 1 \geq 1$ and α a sequence of points in \mathbb{R}^n satisfying (C_i) for every $i \geq 2$. Then we have*

- i) $\mu\dim S_m^n(\alpha) = m$ and $\dim S_m^n(\alpha) = \min\{2m, n-1\}$.
- ii) $\mu\dim(S_m^n(\alpha) \cap H) = m$ for every hyperplane H in \mathbb{R}^n in case $m > 0$.

Proof of Theorem 1. Let α be an arbitrary sequence of points in \mathbb{R}^n which satisfies (C_i) for all $i \geq 2$. We choose a point $q_{i,s}$ from each hyperplane $B_{i,s}^{n-1}$ so that $Q = \{q_{i,s} : i, s \in \mathbb{N}\}$ is discrete in \mathbb{R}^n . Then we define

$$X_m^n = \begin{cases} S_m^n(\alpha) & (0 < m \leq n-1) \\ S_0^n(\alpha) \cup Q & (m = 0) \end{cases}$$

Obviously we have $\dim X_0^n = 0$ because $\dim S_0^n(\alpha) = \dim Q = 0$ and Q is closed. Then it is evident that X_m^n satisfies all of the required conditions in view of Lemma 3. \square

Let N_k^n be the space of those points in \mathbb{R}^n at most k of whose coordinates are rationals. It is known that $\dim N_k^n = k$ (cf. [1]) and $\mu\dim N_k^n = k$ because N_k^n contains a k -simplex. Moreover for every hyperplane H in \mathbb{R}^n , we have

$$(3) \quad k-1 \leq \mu\dim(N_k^n \cap H) \leq \dim(N_k^n \cap H) \leq k, \quad 0 \leq k \leq n.$$

Also it is obvious that $N_m^n \subset S_m^n(\alpha)$ for every sequence α of points in \mathbb{Q}^n . Let A_i^{n-k-1} be the $(n-k-1)$ -planes such that

$$(4) \quad N_k^n = \mathbb{R}^n - \cup \{A_i^{n-k-1} : i \in \mathbb{N}\}, \quad 0 \leq k \leq n-1.$$

We denote by $\mathcal{H}_0 = \{H_i : i \in \mathbb{N}\}$ the family of all hyperplanes in \mathbb{R}^n which are determined by points in \mathbb{Q}^n . Moreover we set $\mathcal{A}_H^{n-k-1} = \{A_i^{n-k-1} : A_i^{n-k-1} \subset H\}$ for arbitrary hyperplanes H in \mathbb{R}^n . Since every non-empty open set in A_i^{n-k-1} contains points in \mathbb{Q}^n densely, we have

Lemma 4. *If U is a non-empty open set in a hyperplane H in \mathbb{R}^n such that $U \cap (\cup \mathcal{A}_H^{n-k-1})$ is dense in U for some k with $0 \leq k \leq n-1$, then $H \in \mathcal{H}_0$.*

Theorem 2. *Let n and k be integers such that $0 \leq k \leq n-1 \geq 1$. Then there exists a space Y_k^n in \mathbb{R}^n such that*

- i) $N_k^n \subset Y_k^n \subset N_{k+1}^n$,
- ii) $\mu\dim Y_k^n = \dim Y_k^n = k$, and
- iii) $\mu\dim(Y_k^n \cap H) = \dim(Y_k^n \cap H) = k$ for every hyperplane H in \mathbb{R}^n .

Proof. First we choose a sequence $\{z_i\}$ of points in \mathbb{Z}^n such that $H_i \cap \text{Int}(z_i + \mathbb{I}^n) \neq \emptyset$ and $\{z_i + \mathbb{I}^n : i \in \mathbb{N}\}$ is discrete in \mathbb{R}^n . Then we can take a k -simplex $\sigma_i^k \subset N_{k+1}^n \cap (z_i + \mathbb{I}^n) \cap H_i$ for every i . We set $Y_k^n = N_k^n \cup \cup \{\sigma_i^k : i \in \mathbb{N}\}$. Then obviously i) and ii) are satisfied. To prove iii), let H be an arbitrary hyperplane with $H \notin \mathcal{H}_0$. Then by Lemma 4, $\cup \mathcal{A}_H^{n-k-1}$ is not dense in H . Hence there exists a non-empty open set U in H such that $U \cap (\cup \mathcal{A}_H^{n-k-1}) = \emptyset$. Then

it is clear that $N_k^n \cap U$ contains a k -simplex and hence $\mu\dim(Y_k^n \cap H) = \dim(Y_k^n \cap H) = k$. \square

In the above proof, it has been proved that

$$(5) \quad \mu\dim(N_k^n \cap H) = \dim(N_k^n \cap H)$$

holds if $H \notin \mathcal{H}_0$; however, as shown in the following, the condition $H \notin \mathcal{H}_0$ can be dropped.

Theorem 3. *For every hyperplane H in \mathbb{R}^n , $\mu\dim(N_k^n \cap H) = \dim(N_k^n \cap H)$, $0 \leq k \leq n-1$.*

Proof. Let H be the hyperplane defined by $\sum_{i=1}^n a_i x_i = b$. We set $\lambda = \{i : a_i \neq 0\}$; here we may assume $n \in \lambda$. If in particular, $\lambda = \{n\}$, then $N_k^n \cap H$ is a copy of N_{k-1}^{n-1} or N_k^{n-1} according as $b/a_n \in \mathbb{Q}$ or not, and (5) follows. Hence we can assume that $s = |\lambda| \geq 2$. Then we claim that

$$(6) \quad \dim(N_k^n \cap H) = \mu\dim(N_k^n \cap H) = k.$$

Let $\pi_0 : \mathbb{R}^n \rightarrow \mathbb{R}^s = \prod \{\mathbb{R}_i : i \in \lambda\}$, $\mathbb{R}_i = \mathbb{R}$, be the projection. Also by $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ we denote the projection defined by $\pi(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$. Then it is obvious

(7) $\pi|_H : H \rightarrow \mathbb{R}^{n-1}$ is a uniform isomorphism, and

(8) if $A_i^{n-k-1} \subset H$, then $A_i^{n-k-1} \subset \pi_0^{-1}(p) \subset H$ for some $p \in \mathbb{Q}^s \cap \pi_0(H)$.

Let us set

$$\mathcal{A} = \{\pi(\pi_0^{-1}(p)) \cap \mathbb{I}^{n-1} : p \in \pi_0(H) \cap \mathbb{Q}^s\} \cup \{\pi(A_i^{n-k-1} \cap H) \cap \mathbb{I}^{n-1} : A_i^{n-k-1} \not\subset H\},$$

where \mathbb{I}^{n-1} is the $(n-1)$ -cube in \mathbb{R}^{n-1} . Then \mathcal{A} is a countable family of closed sets in \mathbb{I}^{n-1} and clearly we have

$$\dim(\pi(\pi_0^{-1}(p)) \cap \mathbb{I}^{n-1}) \leq n-s \leq n-2 \quad \text{for } p \in \pi_0(H) \cap \mathbb{Q}^s, \text{ and}$$

$$\dim(\pi(A_i^{n-k-1} \cap H) \cap \mathbb{I}^{n-1}) \leq n-k-2 \quad \text{if } A_i^{n-k-1} \not\subset H.$$

Moreover it is impossible that $\pi(\pi_0^{-1}(p)) \cap \mathbb{I}^{n-1}$ intersects with all of the $(n-2)$ -faces of \mathbb{I}^{n-1} ; indeed, $\pi(\pi_0^{-1}(p))$ is parallel to at least one of the $(n-2)$ -faces. The same is true as for $\pi(A_i^{n-k-1} \cap H) \cap \mathbb{I}^{n-1}$. Also the following are valid:

$$\pi(\pi_0^{-1}(p)) \cap \pi(\pi_0^{-1}(q)) = \emptyset \quad \text{if } p \neq q \quad (p, q \in \mathbb{Q}^s \cap \pi_0(H)).$$

$$\dim(\pi(A_i^{n-k-1} \cap H) \cap \pi(A_j^{n-k-1} \cap H)) \leq n - k - 3 \text{ if } A_i^{n-k-1} \cap H \neq A_j^{n-k-1} \cap H \\ (A_i^{n-k-1} \not\subset H \text{ and } A_j^{n-k-1} \not\subset H).$$

$$\dim(\pi(\pi_0^{-1}(p)) \cap \pi(A_i^{n-k-1} \cap H)) \leq n - k - 3 \text{ if } A_i^{n-k-1} \cap H \not\subset \pi_0^{-1}(p) \\ (p \in \mathbf{Q}^s \cap \pi_0(H) \text{ and } A_i^{n-k-1} \not\subset H).$$

Hence by [5, Theorem 2] we obtain

$$\mu\dim(\mathbb{I}^{n-1} - \cup \mathcal{A}) \geq n - 1 - (n - k - 3) - 2 = k.$$

Since $\mathbb{I}^{n-1} - \cup \mathcal{A} \subset \pi(N_k^n \cap H)$ by (8), we have $\mu\dim(N_k^n \cap H) = \mu\dim \pi(N_k^n \cap H) \geq k$ by virtue of (7), which proves (6). \square

4. Proof of Main theorem

Hereafter we fix integers n, m and k satisfying the following:

$$0 \leq m \leq n - 1 \geq 1 \text{ and } m \leq k \leq \min\{2m, n - 1\}.$$

Lemma 5(cf.[3]). *Let α be a sequence of points in \mathbb{R}^n . Then*

- i) $\mu\dim(S_m^n(\alpha) \cap N_k^n) = m$, and
- ii) *if in particular, α satisfies the condition (C_i) for all $i \geq 2$, then $\dim(S_m^n(\alpha) \cap N_k^n) = k$.*

Lemma 6. *Let $k \geq m + 1$ and H a hyperplane in \mathbb{R}^n . Then $\mu\dim(S_m^n(\alpha) \cap N_k^n \cap H) = m$ for every α satisfying (C_i) for all $i \geq 2$.*

Henceforth we fix an $\alpha = \{a_i\}$ in \mathbf{Q}^n satisfying (C_i) for all $i \geq 2$. Let us define

$$F_{i,j} = (a_i + F_i^{(n-m-1)}) \cap (a_j + F_j^{(n-m-1)}), \quad i \neq j.$$

In case $2m \leq n - 2$, there exists a collection $\mathcal{F}_{i,j}$ of $(n - 2m - 2)$ -planes such that $\cup \mathcal{F}_{i,j} = F_{i,j}$, and we set $\mathcal{F}_{i,j} = \emptyset$ if $2m \geq n - 1$.

Lemma 7. *For every hyperplane H in \mathbb{R}^n with $H \notin \mathcal{H}_0$, we have*

$$\dim(S_m^n(\alpha) \cap N_k^n \cap H) = k$$

if either $k \leq n - 2$ or $k = m$.

Lemma 8. $k - 1 \leq \dim(S_m^n(\alpha) \cap N_k^n \cap H) \leq k$ for every hyperplane H in \mathbb{R}^n .

Proof of Main theorem. As in the proof of Theorem 2 we take a sequence $\{z_i\}$ of points in \mathbb{Z}^n such that $H_i \cap \text{Int}(z_i + \mathbb{I}^n) \neq \emptyset$ and $\{z_i + \mathbb{I}^n : i \in \mathbb{N}\}$ is discrete in \mathbb{R}^n . Then by

Lemma 8 we can define J_i with $\dim J_i = k$ where $J_i = S_m^n(\alpha) \cap N_k^n \cap (z_i + \mathbb{I}^n) \cap H_i$ or $J_i = S_m^n(\alpha) \cap N_{k+1}^n \cap (z_i + \mathbb{I}^n) \cap H_i$. We set

$$X_{m,k}^n = \begin{cases} (S_m^n(\alpha) \cap N_k^n) \cup \cup \{J_i : i \in \mathbb{N}\} & (m+1 \leq k \leq n-2) \\ S_m^n(\alpha) \cap N_{n-1}^n & (m+1 \leq k = n-1) \\ Y_k^n & (m = k \leq n-1) \end{cases}$$

where $Y_k^n = N_k^n \cup \cup \{\sigma_i^k : i \in \mathbb{N}\}$ and each σ_i^k is a k -simplex contained in $N_{k+1}^n \cap H_i \cap (z_i + \mathbb{I}^n)$ (cf. Theorem 2); here it is possible to choose σ_i^k so that $\sigma_i^k \subset S_k^n(\alpha) \cap N_{k+1}^n \cap H_i \cap (z_i + \mathbb{I}^n)$ (cf. Lemma 8). Then we have

$$S_m^n(\alpha) \cap N_k^n \subset X_{m,k}^n \subset S_m^n(\alpha) \cap N_{k+1}^n,$$

which implies $\mu\dim X_{m,k}^n = k$ by Lemma 5. Also we have $\dim X_{m,k}^n = k$ by Lemma 5 and Theorem 2; we note that each J_i is closed and $\{J_i : i \in \mathbb{N}\}$ is discrete in $X_{m,k}^n$ in case $m+1 \leq k \leq n-2$. Thus the condition i) of Main theorem is satisfied. Moreover the remaining conditions ii) and iii) follow from Lemma 6, Lemma 7 and Theorem 2 directly. This completes the proof of Main theorem. \square

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